THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050B Mathematical Analysis I (Fall 2016)
Tutorial Questions for 3 Nov

We will adopt the following notations:

Let $A \subseteq \mathbb{R}$ be nonempty, $f : A \to \mathbb{R}$ be a function, and $c \in \mathbb{R}$ be a cluster point of A.

1. (a) Define $\lim_{x\to c} f(x) = \infty$.

   (b) Show that $\lim_{x\to 0} \frac{1}{x^2} = \infty$ but it is not true that $\lim_{x\to 0} \frac{1}{x} = \infty$. (Of course, the latter limit does not exist in $\mathbb{R}$ either).

2. (a) Define left and right limit of functions.

   (b) Show by definition that

   $$\lim_{h\to 0^+} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{1}{2\sqrt{x}},$$

   where $x > 0$.

   (c) (Optional) Let $f : [a, b] \to \mathbb{R}$ be increasing, that is, $f(a) \le f(x) \le f(y) \le f(b)$ for any $a \le x \le y \le b$. Show that $\lim_{x\to t^+} f(x)$ exists in $\mathbb{R}$, for any $t \in [a, b)$.

3. (a) Define $\lim_{x\to\infty} f(x) = l \in \mathbb{R}$ and $\lim_{x\to\infty} f(x) = \infty$.

   (b) Show by definition that

   i.
   $$\lim_{x\to\infty} \frac{x^2+1}{x^2+2} = 1$$

   ii.
   $$\lim_{x\to-\infty} -x^2 = -\infty$$

   iii.
   $$\lim_{x\to\infty} (\sqrt{x+1} - \sqrt{x}) = 0$$

4. Let $f : [a, b] \to [s, t]$, $g : [s, t] \to \mathbb{R}$, and $a < c < b$.

   (a) Suppose $\lim_{x\to c} f(x) = l \in \mathbb{R}$, but $f(x) \ne l$ in some δ_0-deleted neighbourhood of c, i.e. there exists $\delta_0 > 0$ such that $f(x) \ne l$ for $0 < |x - l| < \delta_0$. Assume that $\lim_{y\to l} g(y) = M \in \mathbb{R}$. Show that $\lim_{x\to c} g \circ f(x) = M$. You may use the fact that $s \le l \le t$.

   (b) By constructing an example, show that the condition "$f(x) \ne l$ in some δ_0-deleted neighbourhood of c" cannot be removed.

   (c) Alternatively, show that the condition "$f(x) \ne l$ in some δ_0-deleted neighbourhood of c" can be replaced by the continuity of g at l, that is, $\lim_{y\to l} g(y) = g(l)$.